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Twisted quiver bundles over almost complex manifolds

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Abstract

In this paper, we study twisted quiver bundle over general almost complex manifolds. A twisted quiver bundle is a set of *J*-holomorphic vector bundles over an almost complex manifold, labelled by the vertices of a quiver, linked by a set of morphisms twisted by a fixed collection of *J*-holomorphic vector bundles, labelled by the arrows. We prove a Hitchin–Kobayashi correspondence for twisted quiver bundles over a compact almost Hermitian regularized manifold, relating the existence of solutions to certain gauge equations to an appropriate notion of stability for the corresponding quivers. This result can be seen as a generalization of that in [2,9].

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1. Introduction

Let M be a compact Kähler manifold and let E be a holomorphic vector bundle over M. The classical Hitchin–Kobayashi correspondence [10,11,16,19,21,22] states that a holomorphic vector bundle is stable if and only if it is simple (i.e. it admits no non-trivial trace free infinitesimal automorphisms) and admits a Hermitian–Einstein metric.

The Hitchin–Kobayashi correspondence has several interesting and important generalizations and extensions. Quiver bundles, twisted quiver bundles over Kähler manifolds were studied by Alvare-Consul and Garcia-Prada [2,3]. A quiver Q consists of a set Q_0 of vertices $v, v', \ldots$, and a set Q_1 of arrows $a : v \to v'$ connecting the vertices. Given a quiver Q and a compact Kähler manifold M, a *quiver bundle* is defined by assigning a holomorphic vector bundle E_v to a finite number of vertices and a homomorphism $\phi_a : E_v \to E_{v'}$ to a finite number of arrows. A *quiver sheaf* is defined by replacing the term "holomorphic vector bundle" by "coherent sheaf" in the above definition. If we fix a collection of holomorphic vector bundles $\tilde{E}_a$ parametrized by the set of arrows, and the morphisms are $\phi_a : E_v \otimes \tilde{E}_a \to E_{v'}$, twisted by the corresponding bundles, we have a *twisted quiver bundle* or a *twisted quiver sheaf*. In [2] Alvare-Consul and Garcia-Prada defined natural gauge-theoretic equations, *quiver vortex equations,* for a collection of Hermitian metrics on the bundles associated to the vertices of a twisted quiver bundle. To solve these equations, they introduced a stability criterion for twisted quiver sheaves, and proved a Hitchin–Kobayashi correspondence, relating the existence of Hermitian metrics satisfying the quiver vortex equations to the stability bundle. The above result generalized many known results for bundles with extra structure. For examples: *Higgs bundles* [14,20], *holomorphic pair* [6,7], *holomorphic triple, holomorphic chain* [1,8,12]. It should be pointed out Alvare-Consul and Garcia-Prada's results [2,3] cannot be derived from the general Hitchin–Kobayashi correspondence scheme developed by Banfield [5] and further generalized by Mundet i Riera [18]. This is due not only to the presence of twisting vector bundles, but also to the deformation of the Hermitian–Einstein terms in the equations.

In [9], de Bartolomeis and Tian investigated the stability of complex vector bundles over almost complex manifolds, they introduced the concept of bundle almost complex structure (bacs) J on principal bundle, defined J-stable complex vector bundles, and proved the existence of Hermitian–Einstein metrics on J-stable complex vector bundles over a compact almost Hermitian regularized manifold. Inspired by this, we want to discuss twisted quiver bundles over more general almost Hermitian manifolds. In Sections 2 and 3, similar as the appropriate notions of stability and gauge theoretic equations for twisted quiver bundles were defined by Alvare-Consul and Garcia-Prada [2], we will give the definitions of *J-holomorphic twisted quiver bundles,* the *quiver* (σ, τ)*-vortex equations* over almost Hermitian manifolds, the (σ, τ)-degree and the (σ, τ)-stability of J-holomorphic twisted quiver bundle, where σ and τ are collections of real numbers σ_v, τ_v, with σ_v positive for each $v \in Q_0$. Our main result is the following Hitchin–Kobayashi correspondence.

Main theorem. *Let* $R = (\mathbf{E}, \tilde{\mathbf{E}}, Q, J, \phi)$ *be a J-holomorphic twisted quiver bundles on a compact almost Hermitian regularized manifold* (M, η) *(i.e., whose Kähler form* η *satisfies* $\partial\bar{\partial}\eta^{m-1} = 0$*). Let* σ *and* τ *be collections of real numbers* σ_v, τ_v, *with* σ_v *positive for each*

$v \in Q_0$, *such that* $\deg_{\sigma,\tau}(R) = 0$. *Then R is* (σ, τ)*-polystable if and only if it admits a Hermitian metric* **H** *satisfying the quiver* (σ, τ)*-vortex Eqs.* (2.7).

The above theorem combines the results of de Bartolomeis and Tian [9] with the results of Alvare-Consul and Garcia-Prada [2]. By taking over many definitions and results from Alvarez-Consul and Garcia-Prada's [2], de Bartolomeis and Tian [9], we can use the heat flow method to prove the main theorem. In Kähler case, our proof can also be seen as another proof of the Hitchin–Kobayashi correspondence of Alvarez-Consul and Garcia-Prada in [2]. Recently, Lübke and Teleman [23] proved a very general Hitchin–Kobayashi correspondence on arbitrary compact Hermitian manifolds, but their result has no overlap with our theorem, since their result does not include the almost Hermitian case (i.e. in which the integrability condition on the almost complex structure is relaxed). The paper is organized as follows: in Section 2, we give some basic definitions, in Section 3, we give some estimates and preliminaries which will be used in the proof of main theorem; in Section 4, we introduce the definition of (σ, τ)-stability, and prove that (σ, τ)-stability is a necessary condition for the existence of Hermitian metrics satisfying quiver (σ, τ)-vortex equations (2.7); in Section 5, we give the proof of our main theorem.

2. Notations

In this section, we will recall some definitions in [2,9]. Let (M, J_M) be an m-dimensional almost complex manifold. A complex vector bundle $(E, \hat{J})$ of (complex) rank r over M is a real vector bundle E of rank $2r$ equipped with a section $\hat{J}$ of End(E) such that $\hat{J}^2 = -\mathrm{Id}_E$. We denote the principal GL(r, C)-bundle of complex linear frames on E by $C(E)$, thus E can also be seen as an associate bundle of $C(E)$ with standard fibre C^r. Firstly, we recall the notion of bundle almost complex structure (bacs) which has been investigated by de Bartolomeis and Tian in [9].

Definition 2.1. A bundle almost complex structure (bacs) on $C(E)$ is an almost complex structure J on $C(E)$ such that: (1), the bundle projection $\pi : C(E) \to M$ is (J, J_M)-holomorphic; (2), J induces the standard integrable almost complex structure J_S on the fibres; (3), GL(r, C) acts J-holomorphically on $C(E)$.

Let $B(C(E))$ be the set of bacs on $C(E)$, and $\hat{H}(E)$ be the set of linear differential operators $\bar{\partial}_E : \wedge^{p,q}(E) \to \wedge^{p,q+1}(E)$, satisfying the following $\bar{\partial}$-Leibniz rule:

$$\bar{\partial}_E f\alpha = \bar{\partial}_M f \wedge \alpha + f\bar{\partial}_E \alpha, \tag{2.1}$$

for every $f \in C^\infty(M)$, $\alpha \in \wedge^{p,q}(E)$.

Proposition 2.2 (de Bartolomeis and Tian [9]). *The set* $B(C(E))$ *is in one-to-one correspondence with the set* $\hat{H}(E)$.

From the above proposition, we can see that If a bacs J is assigned on $C(E)$, one can define a linear differential operator $\bar{\partial}_E \in \hat{H}(E)$ in natural way. The following definitions are taken from [9].

Definition 2.3. Let $J \in B(C(E))$. Then a section e of E is said to be J-holomorphic if it satisfies $\overline{\partial}_J e = 0$, where the differential operator $\overline{\partial}_E$ is in correspondence with J.

Definition 2.4. Assume bacs's have been assigned on $C(E_2)$ and $C(E_1)$; a bundle morphism $\phi : E_2 \to E_1$ is said to be J-holomorphic if $\overline{\partial}_{E_2^* \otimes E_1} \phi = 0$.

Definition 2.5. Let $J \in B(C(E))$. Then a complex sub-bundle $E' \subset E$ is said to be a J-holomorphic subbundle if $\overline{\partial}_E$ maps $\wedge^{p,q}(E')$ into $\wedge^{p,q+1}(E')$.

Definition 2.6. Let $J \in B(C(E))$. A connection will be called type (1, 0), if its connection 1-forms on $C(E)$ satisfies: $\omega \in T^{1,0}(C(E), gl(r, C), ad)$.

Let $C_J^{1,0}(C(E))$ be the set of all connection 1-forms in $C(E)$ which are of type(1, 0) with respect to J. Given an $\omega \in C_J^{1,0}(C(E))$, it is easy to check that $D_\omega : T^0(C(E)) \to T^1(C(E))$ splits as $D_\omega = \partial_\omega + \overline{\partial}_J$, also we have the splitting $\nabla = \partial_\nabla + \overline{\partial}_E$ of the induced exterior covariant differential operator; and the (1, 1) part of curvature form is $F_\omega^{1,1} = \overline{\partial}_J \omega$ [9, Propositions 1.8 and 1.9].

Assume a Hermitian metric H is assigned on E and let $U_H(E)$ be the principal $U(r)$-bundle of H-unitary frames on E, we have the following result.

Proposition 2.7 (de Bartolomeis and Tian [9, Proposition 2.1]). *There exists a unique connection on $U_H(E)$ such that its connection 1-form, when extended to a connection form on $C(E)$ is of type (1, 0) with respect to $J \in B(C(E))$; this connection is called the canonical Hermitian connection.*

Let $\hat{H} : C(E) \to GL(r, C)$ be defined as following: If $u = \{e_1, \ldots, e_r, \hat{J}e_1, \ldots, \hat{J}e_r\}$, then $\hat{H}(u) = (H(e_j, e_k) - iH(e_j, \hat{J}e_k))_{1\le j,k\le r}$. Set

$$\omega_H = \hat{H}^{-1}\partial_J \hat{H}, \tag{2.2}$$

it is just the canonical Hermitian connection 1-form correspondence with the metric structure H. Let K be another Hermitian structure on E and let $k = K^{-1}H$, it is easy to check that:

$$\omega_H = \omega_K + k^{-1}\partial_{\omega_K} k, \tag{2.3}$$

$$F_{\omega_H}^{1,1} = F_{\omega_K}^{1,1} + \overline{\partial}_E(k^{-1}\partial_{\omega_K} k). \tag{2.4}$$

We now suppose that the almost complex manifold M has a fixed Hermitian metric, with Kähler form η. The natural operator $\Lambda : \Omega_M^{1,1} \to \Omega_M^0$ is the contraction with η. Choose a local real normal coordinate $(x^1, \ldots, x^{2m})$ centered at the considered point p_0. Let

$$J_M\left(\frac{\partial}{\partial x^\alpha}\right) = J_\alpha^\beta \frac{\partial}{\partial x^\beta}, \quad \alpha, \beta = 1, \ldots, 2m.$$

By calculating directly, we have

$$-\sqrt{-1}\Lambda\overline{\partial}\partial f = \frac{1}{2}\Delta f + \frac{1}{2}\sum J_\alpha^\beta \frac{\partial J_\beta^\gamma}{\partial x^\alpha}\frac{\partial f}{\partial x^\gamma} \tag{2.5}$$

at the considered point p_0. Let $\tilde{\Delta} = -2\sqrt{-1}\Lambda\bar{\partial}\partial$, and $V = J_M(g^{\alpha\beta}(\nabla_{\partial/\partial x^\beta} J_M)(\frac{\partial}{\partial x^\alpha}))$, where $(g^{\alpha\beta})$ is the inverse matrix of the metric matrix in local coordinates. From the above equality, we have

$$\tilde{\Delta} f = \Delta f + \langle V, \nabla f\rangle, \tag{2.6}$$

for any $f \in C^2(M)$. In the Kähler case, by the Kodaira identities, we know that $\tilde{\Delta} = \Delta$.

Definition 2.8 (Auslander et al. [4]). A quiver is a pair of sets $Q = (Q_0, Q_1)$ together with two maps $h, t : Q_1 \to Q_0$. The elements of Q_0 (resp. Q_1) are called the vertices (resp. arrows) of the quiver. For each arrow $a \in Q_1$, the vertex ta (resp. ha) is called the tail (resp. head) of the arrow a. The arrow a is sometimes represented by $a : v \to v'$ when $v = ta$ and $v' = ha$.

Throughout this paper, Q is a quiver, and $\tilde{\mathbf{E}}$ is a collection of finite rank complex vector bundles $\tilde{E}_a$ on M, for each arrow $a \in Q_1$. We assign one fixed bacs J_a on each principal bundles $C(\tilde{E}_a)$. The following definitions are taken over from Alvare-Consul and Garcia-Prada [2].

Definition 2.9 (J-holomorphic twisted quiver bundle). A J-holomorphic twisted quiver bundle on almost complex manifold (M, J_M) is a triple $R = (\mathbf{E}, J, \phi)$, where $\mathbf{E}$ is a collection of complex vector bundles E_v of rank r_v on M, J is a collection of bacs J_v on $C(E_v)$, for each $v \in Q_0$, and ϕ is a collection of morphisms $\phi_a : E_{ta} \otimes \tilde{E}_a \to E_{ha}$, for each $a \in Q_1$, such that $E_v = 0$ for all but finitely many $v \in Q_0$, and $\phi_a = 0$ for all but finitely many $a \in Q_1$.

Throughout this paper, $\tilde{\mathbf{H}}$ is a collection of Hermitian metrics $\tilde{H}_a$ on $\tilde{E}_a$, for each $a \in Q_1$, which we fix once and all. A *Hermitian metric* $\mathbf{H}$ on J-holomorphic twisted quiver bundle $R = (\mathbf{E}, J, \phi)$ is a collection of Hermitian metrics H_v on E_v, for each $v \in Q_0$ with $E_v \neq 0$. To define the gauge equations on R, we note that $\phi_a : E_{ta} \otimes \tilde{E}_a \to E_{ha}$ has a smooth adjoint morphism $\phi_a^{*H_a} : E_{ha} \to E_{ta} \otimes \tilde{E}_a$ with respect to the Hermitian metrics $H_{ta} \otimes \tilde{H}_a$ on $E_{ta} \otimes \tilde{E}_a$, and H_{ha} on E_{ha}, for each $a \in Q_1$, so it make sense to consider the composition $\phi_a \circ \phi_a^{*H_a} : E_{ha} \to E_{ha}$. Moreover, ϕ_a and $\phi_a^{*H_a}$ can be seen as morphisms $\phi_a : E_{ta} \to E_{ha} \otimes \tilde{E}_a^*$ and $\phi_a^{*H_a} : E_{ha} \otimes \tilde{E}_a^* \to E_{ta}$, so $\phi_a^{*H_a} \circ \phi_a : E_{ta} \to E_{ta}$ make sense too.

Definition 2.10. (Quiver vortex equations) Let σ and τ be collections of real numbers σ_v, τ_v, with σ_v positive, for each $v \in Q_0$. A Hermitian metric H on R satisfies the *twisted quiver* (σ, τ)*-vortex equations* if

$$\sigma_v\sqrt{-1}\Lambda F_{H_v}^{1,1} + \sum_{a\in h^{-1}(v)} \phi_a \circ \phi_a^{*H_a} - \sum_{a\in t^{-1}(v)} \phi_a^{*H_a} \circ \phi_a = \tau_v \operatorname{Id}_{E_v}, \tag{2.7}$$

for each $v \in Q_0$ such that $E_v \neq 0$, where $F_{H_v}^{1,1}$ is the $(1, 1)$ part of the curvature of the canonical Hermitian connection correspondence with the metric H_v on the complex vector bundle E_v.

Let $R = (\mathbf{E}, \phi)$ be a twisted quiver bundle on m-dimensional almost Hermitian manifold (M, η), $\mathbf{H} = \{H_v\}_{v\in Q_0}$, $\tilde{\mathbf{H}} = \{\tilde{H}_a\}_{a\in Q_1}$ are Hermitian metrics on quiver bundle R and twisting bundles $\tilde{\mathbf{E}}$, respectively. A_v (A_a) denotes the set of connections, that are compatible with H_v (H_a), on E_v ($\tilde{E}_a$). We fix a connection $\tilde{A}_a \in \mathsf{A}_a$ on $\tilde{E}_a$ for each $a \in Q_1$. Then we consider the following generalized Yang–Mills–Higgs functional.

Definition 2.11. The generalized Yang–Mills–Higgs functional $\mathrm{YMH}_{\sigma,\tau} : \mathsf{A} \times \Omega^0 \to R$ by

$$\mathrm{YMH}_{\sigma,\tau}(A, \phi) = \sum_v \sigma_v \|F_{A_v}\|^2_{L^2} + \sum_a \|D_{A_a}\phi_a\|^2_{L^2} + 2\sum_v \sigma_v^{-1} \times \left\| \sum_{a\in h^{-1}(v)} \phi_a \circ \phi_a^{*H} - \sum_{a\in t^{-1}(v)} \phi_a^{*H} \circ \phi_a - \tau_v \mathrm{Id}_{E_v} \right\|^2_{L^2}, \tag{2.8}$$

where $\mathsf{A} = \times_v \mathsf{A}_v$, A_a is the connection induced by A_{ta}, $\tilde{A}_a$ and A_{ha}, L^2 denotes the L^2-norm in the appropriate space of sections.

We have the following decomposition result for the above Yang–Mills–Higgs functional.

Proposition 2.12. *Suppose that the Kähler form η satisfies $\mathrm{d}\eta^{n-1} = 0$, then*

$$\begin{aligned}\mathrm{YMH}_{\sigma,\tau}(A, \phi) = {} & 4\sum_v \sigma_v \|F^{0,2}_{A_v}\|^2_{L^2} + 4\sum_a \|\bar{\partial}_{A_a}\phi_a\|^2_{L^2} - \|\phi\|_{R,\tilde{E}} \\ & + \sum_v \sigma_v^{-1} \left\| \sigma_v\sqrt{-1}\Lambda F^{1,1}_{A_v} + \sum_{a\in h^{-1}(v)} \phi_a \circ \phi_a^{*} - \sum_{a\in t^{-1}(v)} \phi_a^{*H} \circ \phi_a - \tau_v \, \mathrm{Id}_{E_v} \right\|^2_{L^2} \\ & + \sum_v \tau_v \int_M \sqrt{-1}\,\mathrm{Tr}(F_{A_v}) \wedge \eta^{[m-1]} \\ & + \sum_v \sigma_v \int_M \mathrm{Tr}(F_{A_v} \wedge F_{A_v}) \wedge \eta^{[m-2]}.\end{aligned} \tag{2.9}$$

Here $\eta^{[m]} = \frac{\eta^m}{(m)!}$, $F_A^{0,2}$ is the component of F_A of type $(0, 2)$, and $\|\phi\|_{R,\tilde{E}} = \sum_a \int_M \mathrm{Tr}(\phi_a \circ (\mathrm{Id}_{E_{ta}} \otimes \sqrt{-1}\Lambda F^{1,1}_{\tilde{A}_a}) \circ \phi_a^{*H})$.

Proof. First, note that [9, Proposition 3.1]

$$|F_{A_v}|^2 = |F^{1,1}_{A_v}|^2 + 2|F^{0,2}_{A_v}|^2, \tag{2.10}$$

$$|F^{1,1}_{A_v}|^2\eta^{[m]} = \mathrm{Tr}(F_{A_v} \wedge F_{A_v}) \wedge \eta^{[m-1]} + (2|F^{0,2}_{A_v}|^2 + |\lambda F^{1,1}_{A_v}|^2)\eta^{[m]}. \tag{2.11}$$

Secondly, we notice that the 1, 1 part of the curvature of A_a is given by $F^{1,1}_{A_a} = \partial_{A_a}\bar{\partial}_{A_a} + \bar{\partial}_{A_a}\partial_{A_a}$. So, we have

$$F^{1,1}_{A_a}(\phi_a) = F^{1,1}_{A_{ha}} \circ \phi_a - \phi_a \circ (F^{1,1}_{A_{ta}} \otimes \mathrm{Id}_{\tilde{E}_a} + \mathrm{Id}_{E_{ta}} \otimes F^{1,1}_{\tilde{A}_a}). \tag{2.12}$$

By the condition $\mathrm{d}\eta^{m-1} = 0$, one can show that

$$\int_M (\sqrt{-1}\Lambda F^{1,1}_{A_a}\phi_a, \phi_a)\eta^m = \|\partial_{A_a}\phi_a\|^2_{L^2} - \|\bar{\partial}_{A_a}\phi_a\|^2_{L^2}. \tag{2.13}$$

Using the above equalities, and discussing like that in [2, Proposition 4.1], we have the formula (2.9). □

If M is an almost Kähler manifold (i.e. $\mathrm{d}\eta = 0$), the last two terms in (2.9) do not depend on the connection A. By Chern–Weil theorem, we known that they are determined by the first Chern class and second Chern character of E, respectively. An immediate corollary is the following.

Corollary 2.13. *When M is an almost Kähler manifold, the functional $\mathrm{YMH}_{\sigma,\eta}$ is bounded below by*

$$2\pi \sum_v \tau_v C_1(E_v) - 8\pi^2 \sum_v \sigma_v \,\mathrm{Ch}_2(E_v) - \|\phi\|_{R,\tilde{E}},$$

and this lower bounded is attained at $(A, \phi) \in \mathsf{A} \times \Omega^0$ if and only if

$$F^{0,2}_{A_a} = 0, \qquad \bar{\partial}_{A_a}\phi_a = 0,$$
$$\sigma_v\sqrt{-1}\Lambda F^{1,1}_{A_v} = -\sum_{a\in h^{-1}(v)} \phi_a \circ \phi_a^* + \sum_{a\in t^{-1}(v)} \phi_a^{*H} \circ \phi_a + \tau_v \,\mathrm{Id}_{E_v}, \tag{2.14}$$

for every $v \in Q_0$, $a \in Q_1$.

3. Some preliminaries on quiver vortex equations

Given a J-holomorphic twisted quiver bundle $R = (\mathbf{E}, J, \phi)$ on almost Hermitian manifold (M, J_M, η). By the definition, we can assume that $Q = (Q_0, Q_1)$ is a finite quiver, with $E_v \neq 0$ for $v \in Q_0$, and $\phi_a \neq 0$ for $a \in Q_1$. Unless otherwise stated, $v, v', \ldots$ (resp. $a, a', \ldots$) stand for elements of Q_0 (resp. Q_1), while sums, direct sums and products in $v, v', \ldots$ (resp. $a, a', \ldots$) are over elements of Q_0 (resp. Q_1). The main purpose of this paper is to find a Hermitian metric $\mathbf{H}$ satisfying the *twisted quiver* (σ, τ)-*vortex equations* (2.7). Let $\mathbf{K}$ be the initial Hermitian metric on R. Consider a family Hermitian metrics $\mathbf{H(t)}$ on R with initial metric $\mathbf{H}(0) = \mathbf{K}$. And denote $\mathbf{k}(t)$ be collections of endomorphisms $k_v(t) = K_v^{-1}H_v(t)$ on bundle E_v, for $v \in Q_0$ with $E_v \neq 0$. When there is no confusion, we will omit the parameter t and simply write $\mathbf{H}$, $\mathbf{k}$ for $\mathbf{H}(t)$, $\mathbf{k}(t)$. We consider the following heat equations of (2.7)

$$H_v^{-1}\frac{\partial H_v}{\partial t}=-\frac{2}{\sigma_v}\theta_v, \tag{3.1}$$

for each $v\in Q_0$. Where

$$\theta_v(H)=\sigma_v\sqrt{-1}\Lambda F_{H_v}^{1,1}+\sum_{a\in h^{-1}(v)}\phi_a\circ\phi_a^{*H_a}-\sum_{a\in t^{-1}(v)}\phi_a^{*H_a}\circ\phi_a-\tau_v\,\mathrm{Id}_{E_v}. \tag{3.1'}$$

It is completely equivalent to the following evolution equations

$$\begin{aligned}\frac{\partial k_v}{\partial t}&=-2\sqrt{-1}\Lambda\bar\partial_{E_v}\partial_{K_v}k_v+2\sqrt{-1}\Lambda(\bar\partial_{E_v}k_vk_v^{-1}\partial_{K_v}k_v)-2\sqrt{-1}k_v\Lambda F_{K_v}^{1,1}+\frac{2\tau_v}{\sigma_v}k_v\\&\quad-\frac{2}{\sigma_v}\left\{\sum_{a\in h^{-1}(v)}k_v\circ\phi_v\circ k_{ta}^{-1}\otimes\mathrm{Id}_{\tilde E_a}\circ\phi_a^{*K_a}\circ k_v\right.\\&\qquad\left.-\sum_{a\in t^{-1}(v)}\phi_a^{*K_a}\circ k_{ha}\otimes\mathrm{Id}_{\tilde E_a^*}\circ\phi_a\right\},\end{aligned} \tag{3.2}$$

for each $v\in Q_0$. Where we have used the formula (2.4) and the identities

$$\phi_a^{*H_a}=k_{ta}^{-1}\otimes\mathrm{Id}_{\tilde E_a}\circ\phi_a^{*K_a}\circ k_{ha};\quad\text{or}\quad\phi_a^{*H_a}=k_{ta}^{-1}\circ\phi_a^{*K_a}\circ k_{ha}\otimes\mathrm{Id}_{\tilde E_a^*}. \tag{3.3}$$

We know that the above equations are a nonlinear parabolic system, as in [10], $k_v(t)$ are self-adjoint with respect to H_i for $t>0$ since $k_v(0)=\mathrm{Id}_{E_v}$.

Proposition 3.1. *Let* $\mathbf{H}(t)$ *be a solution of the heat flow* (3.1), *then*

$$\left(\tilde\Delta-\frac{\partial}{\partial t}\right)\Theta^2\geq 0, \tag{3.4}$$

and

$$\left(\tilde\Delta-\frac{\partial}{\partial t}\right)\sum_v\mathrm{Tr}\,\theta_v=0, \tag{3.5}$$

where $\theta_v=\theta_v(H)$ *is defined in formula* (3.1′) and $\Theta^2=\sum_v\frac{1}{\sigma_v}|\theta_v|^2_{H_v}$.

Proof. By calculating directly, we have

$$\begin{aligned}\frac{\partial}{\partial t}\theta_v&=\sigma_v\sqrt{-1}\Lambda\bar\partial_{E_v}\left(\partial_{H_v}\left(H_v^{-1}\frac{\partial H_v}{\partial t}\right)\right)\\&\quad-\sum_{a\in h^{-1}(v)}\left(\phi_a\circ H_{ta}^{-1}\frac{\partial H_{ta}}{\partial t}\otimes\mathrm{Id}_{\tilde E_a}\circ\phi_a^{*H_a}-\phi_a\circ\phi_a^{*H_a}\circ H_v^{-1}\frac{\partial H_v}{\partial t}\right)\\&\quad+\sum_{a\in t^{-1}(v)}\left(H_v^{-1}\frac{\partial H_v}{\partial t}\phi_a^{*H_a}\circ\phi_a-\phi_a^{*H_a}\circ H_{ha}^{-1}\frac{\partial H_{ha}}{\partial t}\otimes\mathrm{Id}_{\tilde E_a^*}\circ\phi_a\right),\end{aligned} \tag{3.6}$$

and

$$\tilde{\Delta}|\theta_v|^2_{H_v} = 2Re\langle -2\sqrt{-1}\Lambda\bar{\partial}_{E_v}\partial_{H_v}\theta_v, \theta_v\rangle_{H_v} + \langle [2\sqrt{-1}\Lambda F^{1,1}_{H_v}, \theta_v], \theta_v\rangle_{H_v} + 2|\partial_{H_v}\theta_v|^2_{H_v} + 2|\bar{\partial}_{E_v}\theta_v|^2_{H_v}. \quad (3.7)$$

Using the above formulas, we have

$$\begin{aligned}\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right)\Theta^2 &= \sum_v \frac{2}{\sigma_v}|\nabla_{H_v}\theta_v|^2_{H_v} \\ &\quad + 2\sum_a \left\{\left|\phi_a^{*H_a}\frac{\theta_{ha}}{\sigma_{ha}}\right|^2_H + \left|\frac{\theta_{ta}}{\sigma_{ta}}\phi_a^{*H_a}\right|^2_H - 2 < \phi_a \circ \frac{\theta_{ta}}{\sigma_{ta}} \otimes \mathrm{Id}_{\tilde{E}_a} \circ \phi_a^{*H_a}, \frac{\theta_{ha}}{\sigma_{ha}} >_H\right\} \\ &\quad + 2\sum_a \left\{\left|\phi_a\frac{\theta_{ta}}{\sigma_{ta}}\right|^2_H + \left|\frac{\theta_{ha}}{\sigma_{ha}}\phi_a\right|^2_H - 2 < \phi_a^{*H_a} \circ \frac{\theta_{ha}}{\sigma_{ha}} \otimes \mathrm{Id}_{\tilde{E}_a^*} \circ \phi_a, \frac{\theta_{ta}}{\sigma_{ta}} >_H\right\} \\ &\geq \sum_v \frac{2}{\sigma_v}|\nabla_{H_v}\theta_v|^2_{H_v} \geq 0. \end{aligned} \quad (3.8)$$

The formula (3.5) can be deduce from (3.6) directly. □

Next, we recall the Donaldson's "distance" on the space of Hermitian metrics as follows.

Definition 3.2. For any two Hermitian metrics H, K on a vector bundle E set

$$\sigma(H, K) = \mathrm{Tr}\, H^{-1}K + \mathrm{Tr}\, K^{-1}H - 2\,\mathrm{rank}\, E. \quad (3.9)$$

It is obvious that $\sigma(H, K) \geq 0$ with equality if and only if $H = K$. The function σ is not quite a metric but it serves almost equally well in our problem. In particular, a sequence of metrics H_t converges to H in the usual C^0 topology if and only if $\mathrm{Sup}_M \sigma(H_t, H) \longrightarrow 0$.

Let $\mathbf{H} = \{H_v\}_{v\in Q_0}$ and $\mathbf{K} = \{K_v\}_{v\in Q_0}$ are two Hermitian metrics on the J-holomorphic twisted quiver bundle $R = (\mathbf{E}, J, \phi)$. We define the Donaldson's distance of two metrics on quiver bundle as the following:

$$\sigma(\mathbf{H}, \mathbf{K}) = \sum_v \sigma_v \sigma(H_v, K_v). \quad (3.10)$$

Denoting $\mathbf{k} = \{k_v\}_{v\in Q_0}$, where $k_v = K_v^{-1}H_v$; applying $-\sqrt{-1}\Lambda$ to (2.4) and taking the trace in the bundle E_v, we have

$$\mathrm{Tr}(\sqrt{-1}k_v(\Lambda F^{1,1}_{H_v} - \Lambda F^{1,1}_{K_v})) = -\tfrac{1}{2}\tilde{\Delta}\,\mathrm{Tr}\, k_v + \mathrm{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_{E_v}k_v k_v^{-1}\partial_{K_v}k_v). \quad (3.11)$$

Let $\mathbf{H}(t)$, $\mathbf{K}(t)$ are two families of Hermitian metrics on the quiver bundle R. Using the above formula, we have

$$\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right)\left(\sum_v \sigma_v \operatorname{Tr} k_v(t)\right)$$
$$= 2\sum_v \sigma_v \operatorname{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_{E_v} k_v k_v^{-1}\partial_{K_v} k_v) + \sum_v \operatorname{Tr}\left(k_v\left(\sigma_v K_v^{-1}\frac{\partial K_v}{\partial t} + 2\theta_v(K)\right)\right)$$
$$- \sum_v \operatorname{Tr}\left(k_v\left(\sigma_v H_v^{-1}\frac{\partial H_v}{\partial t} + 2\theta_v(H)\right)\right)$$
$$+ 2\sum_a \operatorname{Tr}\{\phi_a^{*K_a} \circ \phi_a \circ k_{ta} + k_{ha} \circ \phi_a \circ k_{ta}^{-1} \otimes \operatorname{Id}_{\tilde{E}_a} \circ \phi_a^{*K_a} \circ k_{ha}$$
$$- \phi_a^{*K_a} \circ k_{ha} \otimes \operatorname{Id}_{\tilde{E}_a^*} \circ \phi_a - \phi_a \circ \phi_a^{*K_a} \circ k_{ha}\}, \tag{3.12}$$

and

$$\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right)\left(\sum_v \sigma_v \operatorname{Tr} k_v^{-1}(t)\right)$$
$$= 2\sum_v \sigma_v \operatorname{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_{E_v} k_v^{-1} k_v \partial_{K_v} k_v^{-1})$$
$$+ \sum_v \operatorname{Tr}\left(k_v^{-1}\left(\sigma_v H_v^{-1}\frac{\partial H_v}{\partial t} + 2\theta_v(H)\right)\right)$$
$$- \sum_v \operatorname{Tr}\left(k_v^{-1}\left(\sigma_v K_v^{-1}\frac{\partial K_v}{\partial t} + 2\theta_v(K)\right)\right)$$
$$+ 2\sum_a \operatorname{Tr}\{\phi_a^{*H_a} \circ \phi_a \circ k_{ta}^{-1} + k_{ha}^{-1} \circ \phi_a \circ k_{ta} \otimes \operatorname{Id}_{\tilde{E}_a} \circ \phi_a^{*H_a} \circ k_{ha}^{-1}$$
$$- \phi_a^{*H_a} \circ k_{ha}^{-1} \otimes \operatorname{Id}_{\tilde{E}_a^*} \circ \phi_a - \phi_a \circ \phi_a^{*H_a} \circ k_{ha}^{-1}\}, \tag{3.13}$$

On the other hand, from the positivity of k_v, it is not hard to check that

$$\operatorname{Tr}\{\phi_a^{*K_a} \circ \phi_a \circ k_{ta} + k_{ha} \circ \phi_a \circ k_{ta}^{-1} \otimes \operatorname{Id}_{\tilde{E}_a} \circ \phi_a^{*K_a} \circ k_{ha}$$
$$- \phi_a^{*K_a} \circ k_{ha} \otimes \operatorname{Id}_{\tilde{E}_a^*} \circ \phi_a - \phi_a \circ \phi_a^{*K_a} \circ k_{ha}\} \geq 0, \tag{3.14}$$

and

$$\operatorname{Tr}\{\phi_a^{*H_a} \circ \phi_a \circ k_{ta}^{-1} + k_{ha}^{-1} \circ \phi_a \circ k_{ta} \otimes \operatorname{Id}_{\tilde{E}_a} \circ \phi_a^{*H_a} \circ k_{ha}^{-1}$$
$$- \phi_a^{*H_a} \circ k_{ha}^{-1} \otimes \operatorname{Id}_{\tilde{E}_a^*} \circ \phi_a - \phi_a \circ \phi_a^{*H_a} \circ k_{ha}^{-1}\} \geq 0. \tag{3.15}$$

Using the above formula and the facts [10,21]

$$\operatorname{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_{E_v} k_v k_v^{-1}\partial_{K_v} k_v) \geq 0, \qquad \operatorname{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_{E_v} k_v^{-1} k_v \partial_{H_v} k_v^{-1}) \geq 0, \tag{3.16}$$

we have prove the following proposition.

Proposition 3.3. *Let two $n+1$-tuples* $\mathbf{H}(t)$, $\mathbf{K}(t)$ *are two solutions of the Heat flow* (3.1) *then*

$$\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right)\sigma(\mathbf{H}(t), \mathbf{K}(t)) \geq 0. \tag{3.17}$$

Corollary 3.4. *Let* $\mathbf{H}$ *and* $\mathbf{K}$ *are two Hermitian metrics satisfying the quiver* (σ, τ)*-vortex Eq.* (2.7), *then*:

$$\tilde{\Delta}\sigma(\mathbf{H}, \mathbf{K}) \geq 0. \tag{3.18}$$

Proposition 3.5. *Let* $\mathbf{H}(x,t)$ *and* $\mathbf{K}(x,t)$ *are two families of Hermitian metrics on the quiver bundle R, then*

$$\begin{aligned}&\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right)\lg\left\{\sum_v \sigma_v(\mathrm{Tr}(K_v^{-1}H_v) + \mathrm{Tr}(H_v^{-1}K_v))\right\}\\ &\quad\geq -\sum_v\left(\left|H_v^{-1}\frac{\partial H_v}{\partial t} + \frac{2}{\sigma_v}\theta_v(H)\right|_{H_v} + \left|K_v^{-1}\frac{\partial K_v}{\partial t} + \frac{2}{\sigma_v}\theta_v(K)\right|_{K_v}\right).\end{aligned} \tag{3.19}$$

Proof. Let $k_v = K_v^{-1}H_v$, and denote that $A = \sum_v \sigma_v(\mathrm{Tr}\, k_v + \mathrm{Tr}\, k_v^{-1})$. From formula (3.12), (3.13), we have

$$\begin{aligned}&\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right)\lg\{A\}\\ &= A^{-1}\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right)\{A\} - A^{-2}|\nabla A|^2\\ &= A^{-1}\left\{\sum_v \mathrm{Tr}(k_v - k_v^{-1})\left[\left(\sigma_v K_v^{-1}\frac{\partial K_v}{\partial t} + 2\theta_v(K)\right)\right.\right.\\ &\quad\left.\left. - \left(\sigma_v H_v^{-1}\frac{\partial H_v}{\partial t} + 2\theta_v(H)\right)\right]\right\}\\ &\quad + A^{-1}\left\{\sum_v \sigma_v \mathrm{Tr}(-2\sqrt{-1}\Lambda\bar{\partial}_{E_v}k_v k_v^{-1}\partial_{K_v}k_v - 2\sqrt{-1}\Lambda\bar{\partial}_{E_v}k_v^{-1}k_v\partial_{H_v}k_v^{-1})\right\}\\ &\quad - A^{-2}|\nabla A|^2 + 2A^{-1}\sum_a \mathrm{Tr}\{\phi_a^{*K_a}\circ\phi_a\circ k_{ta}\\ &\quad + k_{ha}\circ\phi_a\circ k_{ta}^{-1}\otimes\mathrm{Id}_{\tilde{E}_a}\circ\phi_a^{*K_a}\circ k_{ha} - \phi_a^{*K_a}\circ k_{ha}\otimes\mathrm{Id}_{\tilde{E}_a^*}\circ\phi_a\\ &\quad - \phi_a\circ\phi_a^{*K_a}\circ k_{ha}\} + 2A^{-1}\sum_a \mathrm{Tr}\{\phi_a^{*H_a}\circ\phi_a\circ k_{ta}^{-1} + k_{ha}^{-1}\circ\phi_a\circ k_{ta}\\ &\quad \otimes\mathrm{Id}_{\tilde{E}_a}\circ\phi_a^{*H_a}\circ k_{ha}^{-1} - \phi_a^{*H_a}\circ k_{ha}^{-1}\otimes\mathrm{Id}_{\tilde{E}_a^*}\circ\phi_a - \phi_a\circ\phi_a^{*H_a}\circ k_{ha}^{-1}\},\end{aligned} \tag{3.20}$$

Direct calculation shows that [21]

$$2(\mathrm{Tr}\, k_v)^{-1}\,\mathrm{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_{E_v}k_v k_v^{-1}\partial_{K_v}k_v) - (\mathrm{Tr}\, k_v)^{-2}|\nabla\,\mathrm{Tr}\, k_v|^2 \geq 0,$$
$$2(\mathrm{Tr}\, k_v^{-1})^{-1}\,\mathrm{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_{E_v}k_v^{-1}k_v\partial_{H_v}k_v^{-1}) - (\mathrm{Tr}\, k_v^{-1})^{-2}|\nabla\,\mathrm{Tr}\, k_v^{-1}|^2 \geq 0. \tag{3.21}$$

From the above inequalities, it is easy to check

$$A\left\{\sum_v \sigma_v\,\mathrm{Tr}(-2\sqrt{-1}\Lambda\bar{\partial}_{E_v}k_v k_v^{-1}\partial_{K_v}k_v - 2\sqrt{-1}\Lambda\bar{\partial}_{E_v}k_v^{-1}k_v\partial_{H_v}k_v^{-1})\right\}$$
$$\geq \left|\sum_v \sigma_v(\nabla\,\mathrm{Tr}\, k_v + \nabla\,\mathrm{Tr}\, k_v^{-1})\right|^2. \tag{3.22}$$

By formula (3.14), (3.15), and (3.22), we have proved (3.19). □

Corollary 3.6. *Let* $\mathbf{H}(t)$ *be a solution of the heat flow* (3.1) *with initial metric* $\mathbf{K}$*, then*

$$\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right)\lg\left\{\sum_v \sigma_v(\mathrm{Tr}(K_v^{-1}H_v) + \mathrm{Tr}(H_v^{-1}K_v))\right\} \geq -\sum_v \left|\frac{2}{\sigma_v}\theta_v(K)\right|_{K_v}. \tag{3.23}$$

Corollary 3.7. *Let* $\mathbf{H}$ *and* $\mathbf{K}$ *are two Hermitian metrics on the quiver bundle R, then*

$$\tilde{\Delta}\lg\left\{\sum_v \sigma_v(\mathrm{Tr}(K_v^{-1}H_v) + \mathrm{Tr}(H_v^{-1}K_v))\right\}$$
$$\geq -\sum_v\left(\left|\frac{2}{\sigma_v}\theta_v(H)\right|_{H_v} + \left|\frac{2}{\sigma_v}\theta_v(K)\right|_{K_v}\right). \tag{3.24}$$

At the end of this section, we use the Moser-iteration to deduce the following mean-value inequality which will be used in the proof of main theorem. The major geometric-analytic property of M which we are going to use is the Sobolev inequality on the geodesic ball B_R. Namely, for any $\psi \in C_0^\infty(B(R))$, there exists a constant C_s only dependent on the geometry of M around $B(R)$ such that

$$C_s\left(\int_{B(R)} \psi^{4m/(2m-2)}\right)^{(2m-2)/2m} \leq \int_{B(R)} |\nabla\psi|^2. \tag{3.25}$$

Theorem 3.8. *Suppose that nonnegative function f satisfies*

$$\tilde{\Delta} f \geq -B_1 f, \tag{3.26}$$

where B_1 *is a positive constant. Let* $p > 0$*, then there exist constant* B_2 *depending only on* B_1*, p and M such that*

$$\sup_{B(R/2)} f \leq B_2\left(\int_{B(R)} f^p\right)^{1/p}. \tag{3.27}$$

Proof. Setting $0 < r_2 < r_1 \leq R$, and let φ be the cut-off function

$$\varphi(x) = \begin{cases} 1; & x \in B(r_2), \\ 0; & x \in B(R) \setminus B(r_1), \end{cases} \tag{3.28}$$

$0 \leq \varphi(x) \leq 1$ and $|\nabla\varphi| \leq 2(r_1 - r_2)^{-1}$.

Let $q \geq p > 1$. Multiplying with $f^{q-1}\varphi^2$ on both side of (3.26) and integrating by parts we have

$$\begin{aligned}(q-1)\int_{B(R)} f^{q-2}\varphi^2|\nabla f|^2 \\ \leq -2\int_{B(R)}\langle\nabla\varphi, \nabla f\rangle f^{q-1}\varphi + \int_{B(R)}\langle V, \nabla f\rangle f^{q-1}\varphi^2 + B_1\int_{B(R)} f^q\varphi^2.\end{aligned} \tag{3.29}$$

Using Schwartz inequality and Young inequality, we have

$$\int_{B(R)}|\nabla(f^{q/2}\varphi)|^2 \leq \frac{q}{q-2}\int_{B(R)}(|V|^2 + B_1)f^q\varphi^2 + \int_{B(R)} f^q|\nabla\varphi|^2. \tag{3.30}$$

Applying the Sobolev inequality (3.25) to $f^{q/2}\varphi$, we get

$$\left(\int_{B_{r_2}} f^{q(2m/(2m-2))}\right)^{(2m-2)/2m} \leq C(M, p, B_1, |V|)(1 + (r_1 - r_2)^{-2})\int_{B(r_1)} f^q. \tag{3.31}$$

Then, by the standard Moser-iteration argument we deduce (3.27) for $p > 2$. On the other hand a general argument in [17] shows that $p > 0$ case follows from $p > 2$. □

Corollary 3.9. *If nonnegative function f satisfies*

$$\tilde{\Delta} f \geq -B_3, \tag{3.32}$$

then there exists positive constants B_4, B_5 depending only on M and B_3 such that

$$\|f\|_\infty \leq B_4(\|f\|_1 + B_5). \tag{3.33}$$

Proof. Let $f' = f + B_3$, then we have $\tilde{\Delta} f' \geq -f'$. Applying the mean value inequality (3.27) to f', we can easily conclude the inequality (3.33). □

4. Stability of *J*-holomorphic twisted quiver bundle

Let (M, J_M, η) be a compact m-dimensional almost Hermitian manifold whose Kähler form η satisfies $\partial_M\bar{\partial}_M\eta^{m-1} = 0$, and let $R = (\mathbf{E}, J, \phi)$ be a J-holomorphic twisted quiver bundle over M, and $\tilde{\mathbf{E}} = \{\tilde{E}_a\}_{a\in Q_1}$ are the twisting bundles. Let H_v be a Hermitian metric on the bundle E_v, then the degree of E_v is defined as follows:

$$\deg(E_v) = \frac{\sqrt{-1}}{\mathrm{Vol}(M, \eta)}\int_M(\mathrm{Tr}\,\Lambda F_{H_v}^{1,1})\eta^{[m]}, \tag{4.1}$$

where $\eta^{[m]} = \frac{1}{m!}\eta^m$. From the condition on the Kähler form, we known that the above definition is independent of Hermitian metrics on the E_v. Let $E'_v \subset E_v$ be a complex subbundle, using the Hermitian–Codazzi–Mainardi equation, we have the following proposition [9, Proposition 2.4]:

Proposition 4.1. *Let $(E_v, \hat{J}_v, H_v)$ be a Hermitian bundle with a fixed* bacs J_v, *let $E'_v \subset E_v$ be a complex sub-bundle. Then the following facts are equivalent*:

(1) *E'_v is a J_v-holomorphic sub-bundle*;
(2) *the orthogonal projection $\pi_v : E_v \longrightarrow E'_v$ satisfies*

$$(\mathrm{Id} - \pi_v) \circ \bar{\partial}_{E_v^* \otimes E_v} \pi_v = 0. \tag{4.2}$$

For further consideration, let us introduce the following class of objects $\mathsf{F}(E_v, J_v)$ [9]: $E'_v \in \mathsf{F}(E_v, J_v)$ if and only if

(1) *there exists a closed subset $\Sigma_v \subset M$ with* $\mathrm{H}_{2m-4}(\Sigma_v) < +\infty$, such that *$E'_v|_{M\setminus\Sigma_v}$ is a J_v-holomorphic sub-bundle of $E_v|_{M\setminus\Sigma_v}$*;
(2) *for any $x \in \Sigma_v$, and any local J_M-holomorphic curve C through x not contained in Σ_v, $E'_v|_{C-\{x\}}$ extends to C as sub-bundle*.

Where H_s denote the s-dimensional Hausdorff measure. If $E'_v \in \mathsf{F}(E_v, J_v)$, we will call E'_v be a *weakly J_v-holomorphic sub-bundle* of E_v, and Σ_v be the singular set. On the other hand, when $E'_v \in \mathsf{F}(E_v, J_v)$, it is easy to see that the corresponding section $\pi_v : E_v \to E'_v$ of $E_v^* \otimes E_v$ is in $L_1^2(\mathrm{End}(E_v))$. So it is possible to define the degree of E'_v as follows [9]:

$$\deg(E'_v) := \frac{1}{\mathrm{Vol}(M,\eta)} \int_M (\sqrt{-1}\,\mathrm{Tr}\,\pi_v \Lambda F_{H_v}^{1,1} - |\bar{\partial}_{E_v^* \otimes E_v} \pi_v|^2)\eta^{[m]}, \tag{4.3}$$

and the slope, $\mu(E'_v)$, is defined

$$\mu(E'_v) = \frac{\deg(E'_v)}{\mathrm{rank}\, E'_v}, \tag{4.4}$$

where H_v is any Hermitian metric on E_i. By Codazzi–Mainardi equations, if E'_v is regular, it is easy to check that this definition coincides with the one given in (4.1). In the following, we take over some definitions from [2].

Definition 4.2. Let $R = (\mathbf{E}, J, \phi)$ be a J-holomorphic twisted quiver bundle,

(1) A morphism $f : R \to R'$ between two twisted quiver bundle $R = (\mathbf{E}, J, \phi)$ and $R' = (\mathbf{E}', J', \phi')$ with the same quiver Q is given by a collection of morphisms $f_v : E_v \to E'_v$, for each $v \in Q_0$, such that $\phi'_a \circ (f_{ta} \otimes \mathrm{Id}_{\tilde{E}_a}) = f'_{ha} \circ \phi_a$, for each arrow $a \in Q_1$.
(2) A weakly J-holomorphic quiver sub-bundle of R is another twisted quiver bundle $R' = (\mathbf{E}', \phi')$ such that E'_v is a weakly J_v-holomorphic sub-bundle of E_v with singular set Σ_v,and $\phi_a \circ (f_{ta} \otimes \mathrm{Id}_{\tilde{E}_a}) = f_{ha} \circ \phi'_a$ on $M \setminus \Sigma_{ta} \cup \Sigma_{ha}$ for any $a \in Q_1$, where f_v :

$E'_v \to E_v$ are the inclusion morphisms. When $\cup_v \Sigma_v = \emptyset$, we call R' be a J-holomorphic quiver sub-bundle of R.

(3) A J-holomorphic quiver sub-bundle $R' \hookrightarrow R$ is called proper if $0 < \sum_v \operatorname{rank} E'_v < \sum_v \operatorname{rank} E_v$.

(4) A J-holomorphic twisted quiver bundle R is called decomposable if it can be written as a direct sum $R = R^1 \oplus R^2$ of J-holomorphic quiver sub-bundle with $R^1 \neq R$, $R^2 \neq R$. Otherwise, R is called indecomposable.

(5) A J-holomorphic twisted quiver bundle R is called simple if its only J-holomorphic endomorphisms are the multiples $\lambda \, \mathrm{Id}_R$ of the identity endomorphism.

Definition 4.3. Let σ and τ be collections of real numbers σ_v, τ_v, with σ_v positive, for each $v \in Q_0$. The (σ, τ)-degree and (σ, τ)-slope of quiver bundle R are

$$\deg_{\sigma,\tau}(R) = \sum_v (\sigma_v \deg(E_v) - \tau_v \operatorname{rank}(E_v)), \qquad \mu_{\sigma,\tau}(R) = \frac{\deg_{\sigma,\tau}(R)}{\sum_v \sigma_v \operatorname{rank}(E_v)}, \quad (4.5)$$

respectively. We say that the J-holomorphic twisted quiver bundle R is (σ, τ)-(semi) stable if for all proper weakly J-holomorphic quiver bundle R' of R, $\mu_{\sigma,\tau}(R') < (\leq) \mu_{\sigma,\tau}(R)$. A direct sum of (σ, τ)-stable J-holomorphic twisted quiver bundles, all of them with the same (σ, τ)-slope, is called (σ, τ)-polystable.

Suppose that the quiver bundle R admits a Hermitian metric satisfying the quiver (σ, τ)-vortex equations (2.7), then taking traces in (2.7), integrating over (M, η), and summing for $v \in Q_0$, one sees that the parameters σ, τ are constrained by the relation

$$\deg_{\sigma,\tau}(R) = 0. \quad (4.6)$$

As that in [2], we known that the stability condition does not change under the following two kinds of transformation of the parameters. (1) transform the parameters σ, τ, by multiplying a global constant $c > 0$, obtaining $\sigma' = c\sigma$, $\tau' = c\tau$; (2) transform the parameters τ by $\tau'_v = \tau_v + \mathrm{d}\sigma_v$ for some real number d, and let $\sigma' = \sigma$.

Next, we will show that the (σ, τ)-stability is the necessary condition for the existence of solutions of the quiver (σ, τ)-vortex equations (2.7). In fact, we prove the following theorem.

Theorem 4.4. *Let (M, J_M, η) be a compact m-dimensional almost Hermitian manifold whose Kähler form η satisfies $\partial_M \bar{\partial}_M \eta^{m-1} = 0$, and $R = (\mathbf{E}, J, \phi)$ be a J-holomorphic twisted quiver bundle over M. Let σ and τ are collections of real numbers σ_v, τ_v, with σ_v positive, for each $v \in Q_0$; and satisfy $\deg_{\sigma,\tau}(R) = 0$. Suppose that the quiver bundle R admits a Hermitian metric $\mathbf{H}$ satisfying the quiver (σ, τ)-vortex* Eqs. (2.7), *then R must be (σ, τ)-polystable.*

Proof. This result is proved in exactly the same way as in [2, Section 3.2], so here we only sketch the proof. We can assume that R is indecomposable. Assume that $R' = (\mathbf{E}', \phi')$ be a proper weakly J-holomorphic quiver sub-bundle of R. Let π_v be the H_v-orthogonal projection from E_v onto E'_v section of $E_v^* \otimes E_v$, defined outside singular set. Let $\pi'_v = \mathrm{Id}_{E_v} - \pi_v$, we known that π_v and π'_v are all in $L_1^2(\mathrm{End}(E_v))$. Then, one can show that

$$\mathrm{Vol}(M,\eta)\deg_{\sigma,\tau}(R') = -\sum_v \int_M \sigma_v |\bar{\partial}_{E_v}\pi_v|_H^2 - \sum_a \int_M |\phi_a^{\perp}|_H^2, \tag{4.7}$$

where $\phi_a^{\perp} = \pi_{ha} \circ \phi_a \circ (\pi'_{ta} \otimes \mathrm{Id}_{\tilde{E}_a})$. The indecomposability of R implies that either $\bar{\partial}_{E_v}\pi_v \neq 0$ for some v or $\phi_a^{\perp} \neq 0$ for some a, thus $\mu_{\sigma,\tau}(R') < 0$, hence R is (σ, τ)-stable. □

5. Proof of the main theorem

In this section we will use the (σ, τ)-stability to deduce the existence of a Hermitian metric which satisfies the quiver (σ, τ)-vortex equations (2.7). Let $\mathbf{K} = \{K_v\}_{v\in Q_0}$ be the initial Hermitian metric on the J-holomorphic twisted quiver bundle R, then we consider the evolution equation (3.1), where the parameters σ and τ satisfy $\deg_{\sigma,\tau}(R) = 0$. First of all, we will prove that the above equations have a long-time solution $\mathbf{H}(t)$; next, under the assumption of (σ, τ)-stability, we will show that the solution $\mathbf{H}(t)$ converges to a Hermitian metric $\mathbf{H}(\infty)$ which we need.

From formula (3.2), we known that the evolution equations which we considered is a nonlinear strictly parabolic system, so standard parabolic theory gives the short-time existence.

Proposition 5.1. *For sufficiently small* $\epsilon > 0$, *the system* (3.1) *has a smooth solution* $\mathbf{H}(t) = \{H_v(t)\}_{v\in Q_0}$ *defined for* $0 \le t < \epsilon$.

Let $\mathbf{H}(t)$ be a solution of the evolution equations (3.1), and $k_v = K_v^{-1}H_v$, for all $v \in Q_0$. Then

$$\left|\frac{\partial}{\partial t}(\lg \mathrm{Tr}\, k_v)\right| = \left|\frac{\mathrm{Tr}(\frac{\partial k_v}{\partial t})}{\mathrm{Tr}\, k_v}\right| = \left|\frac{\mathrm{Tr}(k_v H_v^{-1}\frac{\partial H_v}{\partial t})}{\mathrm{Tr}\, k_v}\right| = \frac{2}{\sigma_v}|\theta_v(H)|_{H_v}, \tag{5.1}$$

and similarly

$$\left|\frac{\partial}{\partial t}(\lg \mathrm{Tr}\, k_v^{-1})\right| \le \frac{2}{\sigma_v}|\theta_v(H)|_{H_v}, \tag{5.2}$$

where $\theta_v(H)$ is defined in (3.1′).

Theorem 5.2. *Suppose that a smooth solution* $\mathbf{H}(t)$ *to the evolution equations* (3.1) *is defined for* $0 \le t < T$. *Then* $\mathbf{H}(t)$ *converges in* C^0*-topology to some continuous non-degenerate Hermitian metric* $\mathbf{H}(T)$ *as* $t \to T$.

Proof. Given $\epsilon > 0$, by continuity at $t = 0$ we can find a δ such that

$$\sup_M \sigma(\mathbf{H}(t), \mathbf{H}(t')) < \epsilon,$$

for $0 < t, t' < \delta$. Then Proposition 3.3 and the Maximum principle imply that

$$\sup_M \sigma(\mathbf{H}(t), \mathbf{H}(t')) < \epsilon,$$

for all $t, t' > T - \delta$. This implies that the $H_v(t)$ are a uniformly Cauchy sequence and converge to a continuous limiting metric $H_v(T)$, for every $v \in Q_0$. By Proposition 3.1, we known that $|\theta_v(H)|_{H_v}$ are bounded uniformly. Using formulas (5.1) and (5.2), one can conclude that $\sigma(H_v(t), K_v)$ are bounded uniformly, therefore $H_v(T)$ is a non-degenerate Hermitian metric. □

Arguing like that in [10, Lemma 19] or [15, Lemma 4.3.2], one can easily prove the following lemma.

Lemma 5.3. *Let $H(t)$, $0 \le t < T$, be any one-parameter family of Hermitian metrics on complex vector bundle E over almost Hermitian manifold M. If $H(t)$ converges in the C^0 topology to some continuous metric $H(T)$ as $t \to T$, and if $\sup_M |\Lambda F_H^{1,1}|$ is bounded uniformly in t, then $H(t)$ are bounded in $C^{1,\alpha}$ (for $0 < \alpha < 1$) and also bounded in L_2^p (for any $1 < p < \infty$) uniformly in t.*

Theorem 5.4. *Given any initial tuple $\mathbf{K}$ of Hermitian metrics, then the evolution equation (3.1) has a unique solution $\mathbf{H}(t)$ which exists for $0 \le t < \infty$.*

Proof. Proposition 5.1 guarantees that a solution exists for a short time. Suppose that the solution $\mathbf{H}(t)$ exists for $0 \le t < T$. By Theorem 5.2, $\mathbf{H}(t)$ converges in C^0-topology to a non-degenerate continuous limit Hermitian metric $\mathbf{H}(T)$ as $t \to T$. From Proposition 3.1, we known that $|\theta_v(H)|_{H_v}$ is bounded independently of t. On the other hand, $\mathbf{H}(t)$ are uniformly bounded in C^0-topology, so we known that $\sup_M |\Lambda F_{H_v}^{1,1}|_{K_v}^2$ is bounded independently of t, for every $v \in Q_0$. Hence by Lemma 5.3, $H_v(t)$ are bounded in the C^1-topology and also bounded in L_2^p (for any $1 < p < \infty$) uniformly in t. Since the evolution equation (3.2) is quadratic in the first derivative of k_v we can apply Hamilton's method [13] to deduce that $k_v(t) \to k_v(T)$ in C^∞, equivalently, $H_v(t) \to H_v(T)$, for every $v \in Q_0$, and the solution can be continued past T. Then the evolution equation (3.1) has a solution $\mathbf{H}(t)$ define for all times. The uniqueness of solution can be easily deduced from Proposition 3.3 and the maximum principle. □

For the reader's convenience, we first recall the definition of the Donaldson Lagrangian over almost complex manifolds [9]. Let K_v be a fixed Hermitian metric on the bundle E_v, denote

$$S(E_v, K_v) = \{s \in \Omega^0(M, \mathrm{End}(E_v)) | s^{*K_v} = s\},$$
$$L_2^p S_v = \{s \in L_2^p(\mathrm{End}(E_v)) | s^{*K_v} = s\}, \qquad \mathrm{Met}_{2,v}^p = \{K_v \mathrm{e}^{s_v} | s_v \in L_2^p S_v\}. \tag{5.3}$$

The Donaldson's Lagrangian $M_D : \mathrm{Met}^p_{2,v} \times \mathrm{Met}^p_{2,v} \to R$ is given by

$$M_D(K, H) = 2\int_M \langle \log(K^{-1}H), \sqrt{-1}\Lambda F_K^{1,1}\rangle_K + 2\int_M \langle \log(K^{-1}H), \sqrt{-1}\Lambda\bar{\partial}_E(\Psi[\log(K^{-1}H)](\partial_K \log(K^{-1}H)))\rangle_K,$$

where $\Psi(x, y) = \frac{e^{y-x}+(x-y)-1}{(x-y)^2}$. The Donaldson Lagrangian is additive in the sense that [9],

$$M_D(H^1, H^2) + M_D(H^2, H^3) = M_D(H^1, H^3).$$

The modified Donaldson Lagrangian $M_{\phi,\alpha}$ of two Hermitian metrics on the quiver bundle R is given by [2]

$$M_{\sigma,\tau}(\mathbf{K}, \mathbf{H}) = \sum_v \sigma_v M_D(K_v, H_v) + \sum_a \int_M (|\phi_a|^2_H - |\phi_a|^2_K) - \sum_v \tau_v \int_M \mathrm{Tr}(\log(K_v^{-1}H_v)), \tag{5.4}$$

where $\mathbf{K} = \{K_v\}$, $\mathbf{H} = \{H_v\}$, $K_v, H_v \in Met^p_{2,v}$. By direct calculation, one can show the following lemma [2, Lemma 3.3].

Lemma 5.5.

(1) *Let $\mathbf{H}^1, \mathbf{H}^2, \mathbf{H}^3$ be three Hermitian metrics on quiver bundle R, then*

$$M_{\sigma,\tau}(\mathbf{H}^1, \mathbf{H}^3) = M_{\sigma,\tau}(\mathbf{H}^1, \mathbf{H}^2) + M_{\sigma,\tau}(\mathbf{H}^2, \mathbf{H}^3). \tag{5.5}$$

(2) *Let $\mathbf{H}(t)$ be a family of Hermitian metrics on R, then*

$$\frac{\mathrm{d}}{\mathrm{d}t} M_{\sigma,\tau}(\mathbf{H}(0), \mathbf{H}(t)) = \sum_v \int_M \left\langle H_v^{-1}\frac{\mathrm{d}H_v}{\mathrm{d}t}, \theta_v(H(t))\right\rangle_{H_v(t)}. \tag{5.6}$$

For the further argument, we need the following proposition.

Proposition 5.6 (de Bartolomeis and Tian [9, Theorem 0.2]). *Let (M, J_M, g), (N, J_N, h) be two almost Hermitian manifolds with $\dim_R M = 2m$, and assume there exists a bounded closed 2-form α on N such that $\alpha^{1,1} > 0$ uniformly. Let $\sigma : M \longrightarrow N$ be a L^2_1-weakly (J_M, J_N)-holomorphic map. Then there exists a closed subset $\Sigma \subset M$ with $\mathrm{H}_{2m-4}(\Sigma) < +\infty$, such that σ is smooth on $M \setminus \Sigma$; moreover, for any $x \in \Sigma$, any local J_M-holomorphic curve C through x not contained in Σ, $\sigma|_{C-\{x\}}$ extends smoothly to C.*

Proof of the main theorem. Let $\mathbf{H}(t) = \{(H_v(t)\}$ be a solution of Eq. (3.1) with initial metric $\mathbf{K}$, and $\mathbf{k}(t) = \{k_v(t)\}$, where $k_v = K_v^{-1}H_v = \exp(s_v)$ for all $v \in Q_0$. From Corollary 3.6, we have

$$\tilde{\Delta}\lg\left\{\sum_v \sigma_v(\mathrm{Tr}(k_v)+\mathrm{Tr}(k_v^{-1}))\right\} \geq -\sum_v \frac{2}{\sigma_v}(|\theta_v(H)|_{H_v}+|\theta_v(K)|_{K_v}). \tag{5.7}$$

By Proposition 3.1, we known that $\sup_M |\theta_v(H(t))|_{H_v(t)}$ is bounded independently of t. Using Corollary 3.9, there exists two constants B_5 and B_6 such that

$$\left\|\lg\left\{\sum_v \sigma_v(\mathrm{Tr}(k_v)+\mathrm{Tr}(k_v^{-1}))\right\}\right\|_\infty \leq B_5\left(\int_M \lg\left\{\sum_v \sigma_v(\mathrm{Tr}(k_v)+\mathrm{Tr}(k_v^{-1}))\right\}+B_6\right). \tag{5.8}$$

On the other hand, one can check that

$$\lg\left\{\frac{1}{2\sum_v r_v}\sum_v(\mathrm{Tr}\, k_v+\mathrm{Tr}\, k_v^{-1})\right\} \leq \sum_v |s_v|_{K_v} = \sum_v |s_v|_{H_v} \leq \left(\sum_v r_v^{1/2}\right)\lg\sum_v(\mathrm{Tr}\, k_v+\mathrm{Tr}\, k_v^{-1}), \tag{5.9}$$

where $r_v = \mathrm{rank}\, E_v$. So there exist constants $B_7 > 0$, $B_8 > 0$ such that, for every $t \in [0, +\infty)$, we have:

$$\sum_v \|s_v(t)\|_\infty \leq B_7 + B_8\left(\sum_v \|s_v(t)\|_1\right). \tag{5.10}$$

Now, there are two possibilities:

(1) *There exists constant* $B_9 > 0$ *such that, for every* $t \in [0, +\infty)$,

$$\sum_v \|s_v(t)\|_\infty < B_9.$$

(2) $\limsup_{t\to\infty}(\sum_v \|s_v(t)\|_1) = +\infty$.

Assume we are in case (1). Using the condition $\partial_M\bar{\partial}_M\eta^{m-1} = 0$, it is not hard to check that

$$\begin{aligned}
&\int_M \langle s_v, \sqrt{-1}\Lambda\bar{\partial}_E(\Psi[s_v](\partial_{H_v}s_v))\rangle_{H_v}\eta^{[m]} \\
&= \int_M \langle \Phi[s_v](\bar{\partial}_{E_v}s_v), \bar{\partial}_{E_v}s_v\rangle_{H_v}\eta^{[m]} - \sqrt{-1}\int_M \mathrm{Tr}\, s_v H_v^{-1}\overline{\Psi[s_v](\partial_{H_v}s_v)}^{\mathrm{T}} H_v \wedge \partial\eta^{m-1} \\
&= \int_M \langle \Phi[s_v](\bar{\partial}_{E_v}s_v), \bar{\partial}_{E_v}s_v\rangle_{H_v}\eta^{[m]} - \frac{1}{2}\sqrt{-1}\int_M \bar{\partial}(\mathrm{Tr}\, s_v^2)\wedge\partial\eta^{m-1}
\end{aligned}$$

$$= \int_M \langle \Phi[s_v](\bar{\partial}_{E_v} s_v), \bar{\partial}_{E_v} s_v \rangle_{H_v} \eta^{[m]}.$$

where function $\Phi(x, y) = \Psi(y, x)$. By formula (5.4), we have

$$M_{\sigma,\tau}(\mathbf{K}, \mathbf{H}) \geq -\sum_v \int_M |s_v||\sigma_v \sqrt{-1} \Lambda F_{H_v}^{1,1} - \tau_v \, \mathrm{Id}_{E_v}| \eta^{[m]} + 2 \sum_v \int_M \sigma_v \langle \Phi[s_v](\bar{\partial}_{E_v} s_v), \bar{\partial}_{E_v} s_v \rangle_{H_v} \eta^{[m]} + \sum_a \int_M (|\phi_a|_H^2 - |\phi_a|_K^2). \tag{5.11}$$

From $\sum_v \|s_v(t)\|_\infty < B_9$ for every $t \in [0, +\infty)$, it follows that $\Phi \geq B_{10} > 0$ on the range of the $s_v(t)$'s; so that

$$\int_M \langle \Phi[s_v](\bar{\partial}_{E_v} s_v), \bar{\partial}_{E_v} s_v \rangle_{H_v} \eta^{[m]} \geq B_{10} \|\bar{\partial}_{E_v} s_v\|_2^2, \tag{5.12}$$

for every v. On the other hand, the condition $\sum_v \|s_v(t)\|_\infty < B_9$ implies that $\sum_v |\phi_i|_{H(t)}^2$, and $\sum_v |\sigma_v \sqrt{-1} \Lambda F_{H_v}^{1,1} - \tau_v \, \mathrm{Id}_{E_v}|$ are bounded uniformly. Therefore, there exists $B_{11} > 0$ such that, for every $t \in [0, +\infty)$

$$M_{\sigma,\tau}(\mathbf{K}, \mathbf{H}(t)) \geq -B_{11}. \tag{5.13}$$

From (5.6), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} M_{\phi,\tau}(\mathbf{K}, \mathbf{H}(t)) = -\int_M \sum_v \frac{2}{\sigma_v} |\theta_v(H)|_{H_v}^2. \tag{5.14}$$

By (5.11), (5.12) and (5.14), we known that $\|\bar{\partial}_{E_v} s_v\|_2$ and also $\|\bar{\partial}_{E_v} k_v\|_2$ are uniformly bounded for all $v \in Q_0$. Thus, there exits a subsequences $t_j \to +\infty$, such that $k_v(t_j)$ weakly converges to $k_v(\infty)$ in $L_1^2(S_v)$, for all v. By (5.13) and (5.14), we known that $\sum_v |\theta_v(H)|_{H_v}^2(t_j)$ weakly converges to 0 in $L^2(M)$. Then, the standard elliptic regularity implies that $k_v(\infty)$ is smooth and $H_v(\infty) = K_v k_v(\infty)$ satisfy the quiver (σ, τ)-vortex equations (2.7).

By conformal transformations, we can assume that the initial Hermitian metric $\mathbf{K} = \{K_v\}$ satisfies:

$$\sum_v \mathrm{Tr}(\theta_v(K)) = 0. \tag{5.15}$$

For simplicity, we take over the following notation from [2]. Let $E = \oplus_v E_v$, then, $K = \oplus_v K_v$, and $H = \oplus_v H_v$ are two Hermitian metrics on bundle E, $k = \oplus_v k_v$, and $s = \oplus_v s_v \in S(E, K)$. The morphisms $\phi_a : E_{ta} \otimes \tilde{E}_a \to E_{ha}$ induce a section $\phi = \otimes_a \phi_a$ of the bundle $\Upsilon = \oplus_a \mathrm{Hom}(E_{ta} \otimes \tilde{E}_a, E_{ha})$. H defines a Hermitian metric on Υ, which we shall also denote H, by $(\phi, \phi')_H = \sum_a (\phi_a, \phi'_a)_{H_a}$, where ϕ and ϕ' are two sections of Υ. Given a vector bundle Ξ, we define the endomorphisms $\sigma : \Xi \otimes S^c \to \Xi \otimes S^c$, where $S^c = \oplus_v \mathrm{End}(E_v)$, by fibrewise multiplication, i.e. $(\sigma(f \otimes s))_v = f \otimes \sigma_v s_v$. Given metric H and sections ϕ, ϕ' of bundle Υ, we define the endomorphisms $\phi \circ \phi'^{*H}$, $\phi^{*H} \circ \phi'$, $[\phi, \phi'^{*H}] \in \Omega^0(S^c)$ as follows:

$$(\phi \circ \phi'^{*H})_v = \sum_{a \in h^v} \phi_a \circ \phi_a'^{*H}; \qquad (\phi^{*H} \circ \phi')_v = \sum_{a \in t^{-1}(v)} \phi_a^{*H} \circ \phi_a';$$

$$[\phi, \phi'^{*H}] = \phi \circ \phi'^{*H} - \phi^{*H} \circ \phi'.$$

The quiver (σ, τ)-vortex equations (2.7) can now be written in a compact form.

$$\sigma \circ \sqrt{-1}\Lambda F_H^{1,1} + [\phi, \phi^{*H}] = \tau \circ \mathrm{Id}_E. \tag{5.16}$$

If $H = K\,\mathrm{e}^s \in \mathrm{Met}_2^p$, $\Psi'(x, y) = \mathrm{e}^{x-y}$, then we have [2, Lemma 3.2]

$$|\phi|_H^2 = \sum_a |\phi_a|_{H_a}^2 = \langle \Psi'(s_a)\phi_a, \phi_a\rangle_{K_a} = \langle \Psi'(s)\phi, \phi(s)\rangle_K. \tag{5.17}$$

Assume, from now on, we are in case (2). In particular, we can choose a sequence $\{t_j\}_{j=1}^{\infty}$ such that: $t_j \to \infty$ and $\sum_v \|s_v(t_j)\|_1 \to \infty$. Let $l_j = \|s(t_j)\|_1$ and $u_j = l_j^{-1}s(t_j) \in S(E, K)$, from the assumption, we known that $l_j \to \infty$. Using (5.10), we have

$$\|u_j\|_1 = 1 \quad \text{and} \quad \|u_j\|_\infty \le B_{12}, \tag{5.18}$$

where B_{12} is a positive constant. From formula (3.5) and the above initial assumption (5.15), we have

$$\mathrm{Tr}\, s(t) = 0, \tag{5.19}$$

for every $0 \le t < \infty$. From

$$l_j\langle \Phi[l_j u_j](\bar\partial_E u_j), \bar\partial_E u_j\rangle \ge \langle \Phi[u_j](\bar\partial_E u_j), \bar\partial_E u_j\rangle, \tag{5.20}$$

for j sufficiently large, and (5.11), (5.14), it follows that

$$\int_M \langle \Phi[u_j](\bar\partial_E u_j), \bar\partial_E u_j\rangle \eta^{[m]} \le B_{13}.$$

Since u_j is bounded uniformly, so $\Phi \ge C > 0$ on the range of the u_j's. Then, we obtain

$$\|\bar\partial_E u_j\|_2 \le B_{14}, \tag{5.21}$$

where $\bar\partial_E = \oplus_v \bar\partial_{E_v}$. Then, passing to a subsequence, u_j converges weakly to u_∞ in L_1^2; clearly, u_∞ is nontrivial.

If $\zeta, \zeta_\epsilon \in C^\infty(R \times R, R)$ satisfy $\zeta(x, y) \le (x-y)^{-1}$, whenever $x > y$, and $\zeta_\epsilon(x, y) = 0$ whenever $x - y \le \epsilon$, for some fixed $\epsilon > 0$, then similar to [2, Lemma 3.8], we have

$$\begin{aligned}
&(u_\infty, \sigma \circ \sqrt{-1}\Lambda F_K^{1,1} - \tau \circ \mathrm{Id}_E)_{L^2} + (\sigma \circ \zeta[u_\infty](\bar\partial_E u_\infty), \bar\partial_E u_\infty)_{L^2} + (\zeta_\epsilon[u_\infty]\phi, \phi)_{L^2} \\
&\le \lim_{j\to\infty} l_j^{-1}\left\{\int_M \langle s(t_j), \sigma\sqrt{-1}\Lambda F_K^{1,1} - \tau\,\mathrm{Id}_E\rangle_K + \int_M \langle \sigma\Phi[s(t_j)](\bar\partial_E s(t_j)), \bar\partial_E s(t_j)\rangle\right. \\
&\quad \left. + \int_M (|\phi|_{H(t_j)}^2 - |\phi|_K^2)\right\} \le \lim_{j\to\infty} l_j^{-1} M_{\phi,\tau}(\mathbf{K}, \mathbf{H}(t_j)) = 0.
\end{aligned} \tag{5.22}$$

From the above inequality, one can prove the following two lemmas (the proof is similar to [2, Lemmas 3.9 and 3.10], since the relaxation of the integrability condition on the almost complex structure will change nothing in the proof). □

Lemma 5.7. *The eigenvalues of u_∞ are constant almost everywhere. Let the eigenvalues of u_∞ be $\lambda_1, \dots, \lambda_l$. If $\zeta \in C^\infty(R \times R, R)$ satisfies $\zeta(\lambda_i, \lambda_j) = 0$ whenever $\lambda_i > \lambda_j$, then $\zeta[u_\infty](\bar{\partial}_E u_\infty) = 0$. If ζ_ϵ satisfies $\zeta_\epsilon(x, y) = 0$ whenever $x - y \leq \epsilon$, for some fixed $\epsilon > 0$, then $\zeta_\epsilon[u_\infty]\phi = 0$.*

As above, let $\lambda_1, \dots, \lambda_l$ denote the distinct eigenvalues of the u_∞, listed in ascending order. On the other hand, by (5.19), we have $\operatorname{Tr} u_\infty = 0$ almost everywhere. So $l \geq 2$, and not all the eigenvalues of u_∞ are positive.

For $\alpha < l$ define $p_\alpha : R \to R$ to be a smooth positive function such that

$$p_\alpha(x) = \begin{cases} 1 \text{ if } x \leq \lambda_\alpha, \\ 0 \text{ if } x \geq \lambda_{\alpha+1}. \end{cases} \tag{5.23}$$

Define

$$\pi'_\alpha = p_\alpha(u_\infty). \tag{5.24}$$

Lemma 5.8. *Let π'_α be as above for $\alpha < l$, $\pi_v : E \to E_v$ be the canonical projections and $\pi'_{\alpha,v} = \pi'_\alpha \circ \pi_v$. Then*

(1) $\pi'_\alpha \in L^2_1(S(E, K))$;
(2) $\pi'^2_\alpha = \pi'_\alpha = \pi'^{*K}_\alpha$;
(3) $(\mathrm{Id} - \pi'_\alpha)\bar{\partial}_{E^* \otimes E}(\pi'_\alpha) = 0$ *almost everywhere*;
(4) $(\mathrm{Id} - \pi'_{\alpha,ha}) \circ \phi_a \circ (\pi'_{\alpha,ta} \otimes \mathrm{Id}_{\tilde{E}_a}) = 0$ *for each* $a \in Q_1$.

From the above lemma, we known that the π'_α's are L^2_1-weakly J-holomorphic sub-bundles of E and correspond to L^2_1-weakly J-holomorphic maps from (M, J_M, η) to some Grassmann bundle $Gr_p(E)$. If $U \subset M$ is a sufficiently small domain, then π^{-1}_{Gr} can be equipped with a tamed Symplectic structure just by approximating the standard Kähler structure on $U \times Gr_p(C^r)$. Therefore Proposition 5.6 implies that $\pi'_\alpha \in \mathsf{F}(E, J)$. So $\pi'_{\alpha,v}$ represents a weakly J_v-holomorphic sub-bundle $E'_{\alpha,v}$ of (E_v, J_v). From (4) of Lemma 5.8, we known that the inclusions $E'_{\alpha,v} \hookrightarrow E_v$ are compatible with the morphisms ϕ_a. So, we have obtained a sequence of proper weakly J-holomorphic quiver sub-bundles $R'_\alpha = (\mathbf{E}'_\alpha, \phi'_\alpha)$ of $R = (\mathbf{E}, J, \phi)$;

$$R'_0 \hookrightarrow R'_0 \hookrightarrow \cdots R'_l = R. \tag{5.25}$$

We define

$$Q(\sigma, \tau) := \mathrm{Vol}(M, \eta)(\lambda_l \deg_{\sigma,\tau}(R) - \sum_{\alpha=1}^{l-1}(\lambda_{\alpha+1} - \lambda_\alpha)\deg_{\sigma,\tau}(R'_\alpha)). \tag{5.26}$$

Then

$$
\begin{aligned}
Q(\sigma,\tau) &= \int_M \sqrt{-1}\,\mathrm{Tr}\left\{\left(\lambda_l\,\mathrm{Id}_E-\sum_{\alpha=1}^{l-1}(\lambda_{\alpha+1}-\lambda_\alpha)\pi'_\alpha\right)\sigma\circ\Lambda F_K^{1,1}\right\}\\
&\quad+\int_M\sum_{\alpha=1}^{l-1}(\lambda_{\alpha+1}-\lambda_\alpha)|\bar\partial_{E^*\otimes E}\pi'_\alpha|_K^2\\
&\quad-\mathrm{Vol}(M,\eta)\sum_v\tau_v\left(\lambda_l\,\mathrm{rank}\,E_v-\sum_{\alpha=1}^{l-1}(\lambda_{\alpha+1}-\lambda_\alpha)\,\mathrm{rank}\,E'_{\alpha v}\right)\\
&=\int_M\langle u_\infty,\sigma\circ\sqrt{-1}\Lambda F_K^{1,1}-\tau\circ\mathrm{Id}_E\rangle_K+\int_M\sum_{\alpha=1}^{l-1}(\lambda_{\alpha+1}-\lambda_\alpha)|\bar\partial_{E^*\otimes E}\pi'_\alpha|_K^2.
\end{aligned}
\tag{5.27}
$$

Using the result and notation of [7, Lemma 3.12.1],

$$
\begin{aligned}
&\sum_{\alpha=1}^{l-1}(\lambda_{\alpha+1}-\lambda_\alpha)|\bar\partial_{E^*\otimes E}\pi'_\alpha|^2\\
&\quad=\sum_{\alpha=1}^{l-1}(\lambda_{\alpha+1}-\lambda_\alpha)\langle\bar\partial_{E^*\otimes E}\pi'_\alpha,\bar\partial_{E^*\otimes E}\pi'_\alpha\rangle\\
&\quad=\sum_{\alpha=1}^{l-1}(\lambda_{\alpha+1}-\lambda_\alpha)\langle(\delta p_\alpha)^2[u_\infty]\bar\partial_{E^*\otimes E}u_\infty,\bar\partial_{E^*\otimes E}u_\infty\rangle\\
&\quad=\langle\zeta[u_\infty]\bar\partial_{E^*\otimes E}u_\infty,\bar\partial_{E^*\otimes E}u_\infty\rangle.
\end{aligned}
\tag{5.28}
$$

Here $\zeta : R\times R\to R$ is defined by $\zeta=\sum_{\alpha=0}^{l-1}(\lambda_{\alpha+1}-\lambda_\alpha)(\delta p_\alpha)^2$, hence it satisfies the conditions that $\zeta(\lambda,\mu)\le(\lambda-\mu)^{-1}$ for $\lambda>\mu$. Then, we make use of (5.22), (5.25) and (5.26) to deduce that

$$
Q(\sigma,\tau)\le 0. \tag{5.29}
$$

On the other hand, from the definition of the (σ,τ)-stability of the quiver bundle R we deduce that $Q(\sigma,\tau)>0$, thus we get a contradiction. So, we have proved the main theorem.

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